How our methods of writing algebra have evolved: A thread through history

Jack Oliver
Nambour, Qld
<beja@westnet.com.au>

We have not always had our present clever ways of writing algebraic equations and expressions. This paper attempts to trace how our system has developed since the dawn of civilisation. We will look at a few snapshots taken at distinct times to illustrate this progress.

Ancient Egypt and Babylon

Problem 14 of the Moscow papyrus (circa 1850 BC, in Eves, 1983, p. 11):

> You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. [You are required to find the volume.]
> You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4 result 28. You are to take one-third of 6, result 2. You are to take 28 twice, result 56. You will find it right.

Babylonian tablet of about 2600 BC (in Eves, 1983, p. 14):

> 60 is the circumference, 2 is the sagitta, find the chord.
> [The sagitta is the rise from the middle of a chord to the point on the circle immediately above it.]
> Thou double 2 and get 4, dost thou not see? Take 4 from 20, thou gettest 16. Square 20, thou gettest 400. Square 16, thou gettest 256. Take 256 from 400, thou gettest 144. Find the square root of 144. 12, the square root is the chord. Such is the procedure.

An arithmetical example from Babylon (in Boyer, 1989, p. 32.):

> The sum of 2 numbers is $6\frac{1}{2}$ and their product is $7\frac{1}{2}$. What are the numbers?
> - Find half of $6\frac{1}{2}$. Result $3\frac{1}{4}$.
> - Square $3\frac{1}{4}$. Result $10\frac{9}{16}$.
> - Take $7\frac{1}{2}$ from this. Result $3\frac{1}{16}$.

- Find the square root of $3\frac{1}{16}$. Result $1\frac{3}{4}$.
- Add $3\frac{1}{4}$ to $1\frac{3}{4}$. Result 5.
- Take $1\frac{3}{4}$ from $3\frac{1}{4}$. Result $1\frac{1}{2}$.

Then 5 and $1\frac{1}{2}$ are the required numbers.

The above three examples very much represent the style of how arithmetic and geometric problems were solved in the past. What can we see from this?

- The instructions are a sort of recipe for finding a result.
- The problem is specific to one set of given numbers.
- Answers to what is essentially the same problem with different numbers could be easily obtained by substituting different numbers.

It should be said here that both the Egyptians and the Babylonians had, over the years, constructed all types of arithmetical tables. As examples; the Babylonians constructed multiplication tables of which we have many examples on clay tablets, and the Egyptians had, among other tables, tables to help them manipulate their fractions.

We are not talking about short periods of time. The earliest records go back to about 3000 BC when we first see well-developed civilisations in both Egypt and Mesopotamia and there must have been centuries, if not millennia, of slow development before that. From the earliest records, these parts of the world saw a continuity of civilisation for at least another 2500 years. We are talking about lengths of time considerably longer than the whole of our present "Christian era." There would have been ample time to develop all sorts of fascinating techniques to solve mathematical problems. What the records show is a "recipe" or algorithm for solving various problems. The records do not usually give us hints as to how the algorithms were developed.

Greece

We shall say very little about the Greek period from Pythagoras through Euclid to Archimedes and Pappus because it does not add much to our story. This was the period of great Greek geometry when many essentially algebraic problems were investigated in a geometrical form as in the various "Books of Euclid" (Kramer, 1982, p. 370).

We pick up the story again with Diophantus of Alexandria who probably lived in the middle of the third century of our era (around AD 250). He wrote three books:

- *Arithmetica* (6 of 13 books are extant)
- *On Polygonal Numbers* (a fragment exists)
- *Porisms* (lost).

It is the *Arithmetica* that is of interest to us. It is a book on the theory of numbers. The extant part solves about 130 problems which are problems involving what we today would call determinate and indeterminate equations. To give a flavour to the *Arithmetica*, one of the problems (Problem 10, Book IV) is: "Find two numbers such that their sum is equal to the sum of their cubes" (in Eves, 1983, p. 119).

Diophantus looked for positive rational solutions and seemed to be satisfied after he had found a solution, even though other solutions to the same problem might exist.

He introduced an algebraic symbolism using an abbreviation for the unknown (Eves, 1983, p. 317). Up to this time, problems and their solutions had been written out in a prose-like form as indicated in the earlier part of this paper. This, as you can see at the beginning of this paper, is cumbersome and makes it very difficult to understand the method of solution. By introducing abbreviations, it made the work less cumbersome and hopefully easier to understand. Diophantus was very much before his time in using this approach and it must be said that the majority of mathematicians still used the earlier approach for at least the next 1000 years. Still, a start had been made.

It is convenient to divide the development into three periods of time and they have been called by the following names:

- *rhetorical* period — the period before Diophantus when everything had been written out in full;
- *syncopation* period — when some abbreviations had been introduced (roughly AD 250–1600);
- *symbolic* period — when the work is stated entirely in symbols that can be manipulated by a set of rules that have mainly been standardised.

The rest of this paper will be concerned with the developments in the syncopation period that will lead us to the symbolic period.

In solving these problems, Diophantus would have been faced with expressions (written in the terminology of today) such as:

$$2x^3 - 3x^2 + 4x - 5$$

Up to this time, this would have been expressed in words (rhetorical):

"A first number is formed by taking twice the cube of a second number and the quadruple of the second less the triple of the square of the second number and five."

Such expressions must have been extremely difficult to comprehend and manipulate. Diophantus designed his own "shorthand" system and would have written this as:

$$\mathrm{K}^{\gamma}\,\beta\,\zeta\,\delta\,\pitchfork\,\Delta^{\gamma}\,\gamma\,\mathring{\mathrm{M}}\,\varepsilon$$

This, as you can see, is much shorter and, given the rules, would have been much easier to comprehend at a glance. A partial and sufficient key (for our purposes) is given in Table 1.

Table 1

Upper case		***Lower case***	
$\mathring{\mathrm{M}}$	Unit		
ζ	Unknown	β	2
Δ^{γ}	Unknown squared	γ	3
K^{γ}	Unknown cubed	δ	4
$\pitchfork$	Minus	ε	5

Addition is by juxtaposition. The added terms are grouped at the left hand side and the removed (minus) terms are grouped at the right hand side of the expression (Eves, 1983, pp. 128–129).

Lower case Greek letters are used for numbers (the standard Classical Greek system) and upper case Greek letters are used for operations. The

minus is represented by an upper case lambda (Λ) with an extra leg and the unknown by a ζ. This is summarised in Table 2.

Table 2

$K^{\gamma}\beta$	Twice cube unknown
$\zeta\,\delta$	Quadruple of unknown
Λ	Minus
$\Delta^{\gamma}\gamma$	Triple square of unknown
$\overset{\circ}{M}\,\varepsilon$	Five units

Note that there was no symbol for zero in the time of Diophantus.

As mentioned above, Diophantus was very much ahead of his time in trying to introduce an algebraic symbolism. Although there were sporadic attempts to do so from the time of Diophantus until the early years of the European Renaissance no essential further progress was made until the 15th and 16th centuries.

Mention here should be made of the work of Hindu mathematicians in the syncopation of algebra (Eves, 1983, p. 130; Kramer, 1982, p. 66). Among others, Brahmagupta in the early 7th century used a form of syncopation. Other Hindu mathematicians after Brahmagupta used similar forms of syncopation. Table 3 gives a key for the main symbols and operations.

Table 3

Item	***Implementation***
Addition	By juxtaposition
Subtraction	Dot over the subtrahend
Multiplication	bha after factors to be multiplied
Square root	ka
Unknown	yā
Known integer	rū
2nd unknown	kā

ka, yā, rū, kā are transliterations of abbreviations of Sanskrit words

Table 4 shows some simple examples of modern algebraic expressions translated into the notation of Brahmagupta.

Table 4

Modern expression	***Brahmagupta's expression***
$x+8$	yā rū 8
$5xy$	yā ka 5 bha
$\sqrt{4x}$	ka yā 4 bha
$x-7$	yā rū 7

After the time of Diophantus, both the Hindu and Arab mathematicians used syncopated algebraic to help them in their investigations into increasingly complicated problems.

This now brings us to the time of the Renaissance when Europe was becoming the centre of the mathematical world. More advanced mathematics was being looked at and an improved terminology was sorely needed.

The final part of this article looks at the period from about AD 1450 to about AD 1650. During this time, various mathematicians devised shorthands that were progressing from the syncopated towards the symbolic, until by the time of the mid 1600s, we have a system of writing algebra that would be recognisable today. We shall look at "snapshots" taken at various times across this period to show how this happened (Hogben, 1936, p. 259).

Our first example is from Regiomontanus, author of *De Triangulis*, whose real name was Johann Müller of Königsberg (1436–1476) — an important mathematician of his time (Boyer, 1989, p. 272). An example from 1464 is:

3 census et 6 demptis 5 rebus aequatur zero,

which reads, “$3x^2$ and 6 less $5x$ equals 0.”

Thirty years later, Pacioli would have written the same expression in his “Summa de arithmetica” (1494) as:

3 ce p 6 m 5 rebus ae 0

(ce short for census, “square of unknown,” p short for piu, “more”, m short for meno, “less”).

Robert Recorde introduced the sign “=” for the word “equals” in 1557 in his book *The Whetstone of Witte* (Eves, 1983, p. 130).

Simon Stevin (1548–1620) in Flanders in 1585 would have written this as:

$$3\,\textcircled{2} - 5\,\textcircled{1} + 6\,\textcircled{\cdot} = 0$$

or

$$\overset{\textcircled{2}}{3} - \overset{\textcircled{1}}{5} + \overset{\textcircled{\cdot}}{6} = 0$$

While in 1591 in France, François Viète (Franciscus Vieta, 1540–1603) would have written:

3 in A quad – 5 in A plano + 6 aequatur 0.

Viète was one of the greatest mathematicians of all time but his notation here looks very conservative compared with that of Stevin.

Moving into the next century, Descartes would have written this in 1637 (the year of his *Discours de la Méthode* (Boyer, 1989, p. 336)) as:

$$3x^2 - 5x + 6 = 0$$

which is as we would write it today.

Regiomontanus (1464) uses a structure very similar to that of Diophantus. He has a zero symbol, which was not available to Diophantus, but uses complete words rather than abbreviations. Pacioli makes the advance of using some abbreviations. Both Regiomantanus and Pacioli group all the positive terms on the left followed by the grouped negative terms before equating the whole expression to zero. Stevin (1585) uses a structure very similar to our modern structure, as did Viète (1591), including the symbols “+”, “–”, and “=”. Viète used “A” to represent the unknown and would use “B” to represent a second unknown, etc., which is very similar to our modern notation. However, he still used a mixture of complete words and abbreviations. Descartes, at the start of the modern era, used letters at the end of the alphabet to stand for unknowns and letters at the beginning of the alphabet for (generalised) knowns.

At this point we end our story as we enter the modern era during which mathematical notation has been dramatically expanded upon the foundations we have described in the above account.

References

Boyer, C. (1989). *A History of Mathematics* (rev.). New York: Wiley.

Eves, H. (1983). *Great Moments in Mathematics before 1650.* The Mathematical Association of America.

Heath, T. (1908; rev. 1956). *The Thirteen Books of Euclid's Elements (2nd Ed.).* Dover Publications.

Hogben, L. (1936, rev. 1989). Mathematics for the Million. Rendlesham, UK: Merlin Press.

Kramer, E. (1982). *The Nature and Growth of Modern Mathematics, Vol. 1.* Princeton University Press.